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Cost Cumulant-Based Control for a Class of Linear Quadratic Tracking Problems

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***Abstract*— The topic of cost cumulant control is currently receiving substantial research from the theoretical community oriented toward stochastic control theory. For instance, the present paper extends the application of cost cumulant controller design to control of a wide class of linear quadratic tracking systems. It is shown that the tracking problem can be solved in two parts: a feedback k-cost-cumulant (kCC) control whose optimization criterion representing a linear combination of finite k cumulant indices of a finite horizon integral quadratic cost associated to a linear tracking stochastic system is determined by a set of Riccati-type differential equations and a set of time-dependent tracking variables is found by solving an auxiliary set of differential equations (incorporating the desired trajectory) backward from a stable final time.**

I. Preliminaries

An interesting extension of the cost-cumulant control theory [1]-[5] when both perfect and noisy state measurements are available, is to consider following a specified output trajectory as closely as possible in the sense of cost-cumulant control objective. Some motivations for this theoretical development are found in the altitude control of a terrain-following aircraft where there is knowledge of the future terrain; and in tactical and combat situations wherein a vehicle with the goal seeking nature initially decides on an appropriate destination and then moves in an optimal fashion toward that destination. Consider a linear stochastic tracking system governed by

$$dx(t) = (A(t)x(t) + B(t)u(t))dt + G(t)dw(t),\ x(t_0) \quad (1)$$
$$y(t) = C(t)x(t) \quad (2)$$

where the coefficients $A \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times m})$, $C \in \mathcal{C}([t_0,t_f];\mathbb{R}^{r\times n})$, and $G \in \mathcal{C}([t_0,t_f];\mathbb{R}^{n\times p})$. The system noise $w(t) \in \mathbb{R}^p$ is the p-dimensional stationary Wiener process starting from t_0, independent of $x(t_0) = x_0$, and defined on a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation

$$E\left\{[w(\tau)-w(\xi)][w(\tau)-w(\xi)]^T\right\} = W|\tau-\xi|\,, \quad W > 0\,.$$

The control input $u \in L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))$ the subset of Hilbert space of $\mathbb{R}^m$-valued square-integrable process on $[t_0,t_f]$ that are adapted to the σ-field $\mathcal{F}_t$ generated by $w(t)$ to the specified system model is selected so that the resulting output $y \in L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^r))$ best matches the desired output $z \in L^2([t_0,t_f];\mathbb{R}^r)$ in the cost cumulant optimization criterion which will be clear shortly. Associated with the initial condition $(t_0, x_0; u) \in [t_0,t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))$ is a traditional finite-horizon IQF random cost $J : [t_0,t_f] \times \mathbb{R}^n \times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m)) \mapsto \mathbb{R}^+$ such that

$$J(t_0,x_0;u) = [z(t_f)-y(t_f)]^T Q_f\,[z(t_f)-y(t_f)] \quad (3)$$
$$+\int_{t_0}^{t_f}\{[z(\tau)-y(\tau)]^T Q(\tau)\,[z(\tau)-y(\tau)] + u^T(\tau)R(\tau)u(\tau)\}d\tau$$

in which the terminal penalty error weighting $Q_f \in \mathbb{R}^{r\times r}$, the error weighting $Q \in \mathcal{C}([t_0,t_f];\mathbb{R}^{r\times r})$, and the control input weighting $R \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m\times m})$ are symmetric and positive semidefinite with $R(t)$ invertible.

In the perfect-state measurement case, the initial state is assumed to be known exactly, and the control input is generated by a closed-loop control policy of interest $\gamma : [t_0,t_f] \times L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^n)) \mapsto L^2_{\mathcal{F}_t}(\Omega;\mathcal{C}([t_0,t_f];\mathbb{R}^m))$, according to the control law

$$u(t) = \gamma(t, x(t)) = K(t)x(t) + u_{ext}(t)\,, \quad (4)$$

where $u_{ext} \in \mathcal{C}([t_0,t_f];\mathbb{R}^m)$ is an external signal and $K \in \mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n})$ is an admissible feedback gain in a sense to be specified later. Hence, for the given initial condition $(t_0,x_0) \in [t_0,t_f] \times \mathbb{R}^n$ and subject to the control policy (4), the dynamics of the tracking problem are governed by

$$dx(t) = [A(t) + B(t)K(t)]x(t)dt + B(t)u_{ext}(t)dt$$
$$+ G(t)dw(t)\,, \qquad x(t_0) = x_0\,, \quad (5)$$
$$y(t) = C(t)x(t)\,, \quad (6)$$

and the IQF random cost

$$J(t_0,x_0;K,u_{ext}) = [z(t_f)-y(t_f)]^T Q_f\,[z(t_f)-y(t_f)]$$
$$+\int_{t_0}^{t_f}\Big\{[z(\tau)-y(\tau)]^T Q(\tau)\,[z(\tau)-y(\tau)] \quad (7)$$
$$+ [K(\tau)x(\tau)+u_{ext}(\tau)]^T R(\tau)[K(\tau)x(\tau)+u_{ext}(\tau)]\Big\}d\tau.$$

It is now ready to generate some cost cumulants for the finite-horizon tracking problem. These cost statistics are subsequently utilized in defining the performance index arisen in state-feedback kCC control over a finite horizon optimization. In general, it is suggested that the initial condition (t_0,x_0) should be replaced by any arbitrary pair (α, x_α). Then, for the given external signal u_{ext} and admissible feedback gain K, the cost functional (7) is seen as the "cost-to-go", $J(\alpha, x_\alpha)$. The moment-generating function of the

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vector-valued random process (5) is given by the definition

$$\varphi(\alpha, x_\alpha; \theta) = E\{\exp(\theta J(\alpha, x_\alpha))\}\,, \tag{8}$$

where the scalar $\theta \in \mathbb{R}^+$ is a small parameter. Thus, the cumulant-generating function immediately follows

$$\psi(\alpha, x_\alpha; \theta) = \ln\{\varphi(\alpha, x_\alpha; \theta)\}\,, \tag{9}$$

in which $\ln\{\cdot\}$ denotes the natural logarithmic transformation of an enclosed entity.

Theorem 1: For all $\alpha \in [t_0, t_f]$ and the small parameter $\theta \in \mathbb{R}^+$, define $\varphi(\alpha, x_\alpha; \theta) = \varrho(\alpha, \theta) \exp\{x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha + 2x_\alpha^T \eta(\alpha, \theta)\}$ and $\upsilon(\alpha, \theta) = \ln\{\varrho(\alpha, \theta)\}$. Then, the cost cumulant-generating function can be expressed as follows

$$\psi(\alpha, x_\alpha; \theta) = x_\alpha^T \Upsilon(\alpha, \theta) x_\alpha + 2x_\alpha^T \eta(\alpha, \theta) + \upsilon(\alpha, \theta) \tag{10}$$

where $\Upsilon(\alpha, \theta)$, $\eta(\alpha, \theta)$, and $\upsilon(\alpha, \theta)$ solve the backward-in-time differential equations

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \Upsilon(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)[A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - 2\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) \\ &\quad - \theta C^T(\alpha)Q(\alpha)C(\alpha) - \theta K^T(\alpha)R(\alpha)K(\alpha)\,,\end{aligned} \tag{11}$$

$$\begin{aligned}\frac{d}{d\alpha}\eta(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \eta(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)B(\alpha)u_{ext}(\alpha) \\ &\quad - \theta K^T(\alpha)R(\alpha)u_{ext}(\alpha) + \theta C^T(\alpha)Q(\alpha)z(\alpha)\,,\end{aligned} \tag{12}$$

$$\begin{aligned}\frac{d}{d\alpha}\upsilon(\alpha, \theta) &= -\mathrm{Tr}\left\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\right\} \\ &\quad - 2\eta^T(\alpha, \theta)B(\alpha)u_{ext}(\alpha) \\ &\quad - \theta u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) - \theta z^T(\alpha)Q(\alpha)z(\alpha)\end{aligned} \tag{13}$$

with the terminal conditions $\Upsilon(t_f, \theta) = \theta C^T(t_f)Q_f C(t_f)$, $\eta(t_f, \theta) = \theta C^T(t_f)Q_f z(t_f)$, $\upsilon(t_f, \theta) = \theta z^T(t_f)Q_f z(t_f)$.

Proof. For any θ given, let $\varpi(\alpha, x_\alpha; \theta) = \exp\{\theta J(\alpha, x_\alpha)\}$, then the moment-generating function becomes $\varphi(\alpha, x_\alpha; \theta) = E\{\varpi(\alpha, x_\alpha; \theta)\}$ with the time derivative of

$$\begin{aligned}\frac{d}{d\alpha}\varphi(\alpha, x_\alpha; \theta) &= -\theta\Big\{x_\alpha^T[C^T(\alpha)Q(\alpha)C(\alpha) \\ &\quad + K^T(\alpha)R(\alpha)K(\alpha)]x_\alpha + 2x_\alpha^T[K^T(\alpha)R(\alpha)u_{ext}(\alpha) \\ &\quad - C^T(\alpha)Q(\alpha)z(\alpha)] + u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) \\ &\quad + z^T(\alpha)Q(\alpha)z(\alpha)\Big\}\varphi(\alpha, x_\alpha; \theta)\,.\end{aligned} \tag{14}$$

Using the standard Ito's formula, it yields

$$\begin{aligned}d\varphi(\alpha, x_\alpha; \theta) &= E\{d\varpi(\alpha, x_\alpha; \theta)\}\,, \\ &= E\Big\{\varpi_\alpha(\alpha, x_\alpha; \theta)\,d\alpha + \varpi_{x_\alpha}(\alpha, x_\alpha; \theta)\,dx_\alpha \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varpi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta)G(\alpha)WG^T(\alpha)\right\}d\alpha\Big\}\,, \\ &= \varphi_{x_\alpha}(\alpha, x_\alpha; \theta)\,[A(\alpha) + B(\alpha)K(\alpha)]\,x_\alpha d\alpha \\ &\quad + \varphi_\alpha(\alpha, x_\alpha; \theta)\,d\alpha + \varphi_{x_\alpha}(\alpha, x_\alpha; \theta)\,B(\alpha)u_{ext}(\alpha)d\alpha \\ &\quad + \frac{1}{2}\mathrm{Tr}\left\{\varphi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta)\,G(\alpha)\,WG^T(\alpha)\right\}d\alpha\,,\end{aligned}$$

which under the definition $\varphi(\alpha, x_\alpha; \theta) = \varrho(\alpha, \theta)\exp\{x_\alpha^T \Upsilon(\alpha, \theta)x_\alpha + 2x_\alpha^T \eta(\alpha, \theta)\}$ and the partial derivatives

$$\varphi_\alpha(\alpha, x_\alpha; \theta) = \left[\frac{\frac{d}{d\alpha}\varrho(\alpha, \theta)}{\varrho(\alpha, \theta)} + x_\alpha^T \frac{d}{d\alpha}\Upsilon(\alpha, \theta)x_\alpha + 2x_\alpha^T \frac{d}{d\alpha}\eta(\alpha, \theta)\right]\varphi(\alpha, x_\alpha; \theta)$$

$$\varphi_{x_\alpha}(\alpha, x_\alpha; \theta) = \left\{x_\alpha^T\left[\Upsilon(\alpha, \theta) + \Upsilon^T(\alpha, \theta)\right] + 2\eta^T(\alpha, \theta)\right\}\varphi(\alpha, x_\alpha; \theta)\,,$$

$$\begin{aligned}\varphi_{x_\alpha x_\alpha}(\alpha, x_\alpha; \theta) &= \left[\Upsilon(\alpha, \theta) + \Upsilon^T(\alpha, \theta)\right]\varphi(\alpha, x_\alpha; \theta) \\ &+ \left[\Upsilon(\alpha, \theta) + \Upsilon^T(\alpha, \theta)\right]x_\alpha x_\alpha^T\left[\Upsilon(\alpha, \theta) + \Upsilon^T(\alpha, \theta)\right]\varphi(\alpha, x_\alpha; \theta)\end{aligned}$$

leads to

$$\begin{aligned}\frac{d}{d\alpha}\varphi(\alpha, x_\alpha; \theta) &= \frac{\frac{d}{d\alpha}\varrho(\alpha, \theta)}{\varrho(\alpha, \theta)}\varphi(\alpha, x_\alpha; \theta) \\ &+ \left[x_\alpha^T \frac{d}{d\alpha}\Upsilon(\alpha, \theta)x_\alpha + 2x_\alpha^T \frac{d}{d\alpha}\eta(\alpha, \theta)\right]\varphi(\alpha, x_\alpha; \theta) \\ &+ x_\alpha^T[A(\alpha) + B(\alpha)K(\alpha)]^T\Upsilon(\alpha, \theta)x_\alpha\varphi(\alpha, x_\alpha; \theta) \\ &+ x_\alpha^T\Upsilon(\alpha, \theta)[A(\alpha) + B(\alpha)K(\alpha)]x_\alpha\varphi(\alpha, x_\alpha; \theta) \\ &+ 2x_\alpha^T[A(\alpha) + B(\alpha)K(\alpha)]^T\eta(\alpha, \theta)\varphi(\alpha, x_\alpha; \theta) \\ &+ 2x_\alpha^T\Upsilon(\alpha, \theta)B(\alpha)u_{ext}(\alpha)\varphi(\alpha, x_\alpha; \theta) \\ &+ 2\eta^T(\alpha, \theta)B(\alpha)u_{ext}(\alpha)\varphi(\alpha, x_\alpha; \theta) \\ &+ \mathrm{Tr}\left\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\right\}\varphi(\alpha, x_\alpha; \theta) \\ &+ 2x_\alpha^T\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta)x_\alpha\varphi(\alpha, x_\alpha; \theta)\,.\end{aligned} \tag{15}$$

Replacing (14) into (15) and having both linear and quadratic terms independent of x_α, it requires that

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \Upsilon(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)[A(\alpha) + B(\alpha)K(\alpha)] \\ &\quad - 2\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, \theta) \\ &\quad - \theta C^T(\alpha)Q(\alpha)C(\alpha) - \theta K^T(\alpha)R(\alpha)K(\alpha)\,,\end{aligned}$$

$$\begin{aligned}\frac{d}{d\alpha}\eta(\alpha, \theta) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T \eta(\alpha, \theta) \\ &\quad - \Upsilon(\alpha, \theta)B(\alpha)u_{ext}(\alpha) \\ &\quad - \theta K^T(\alpha)R(\alpha)u_{ext}(\alpha) + \theta C^T(\alpha)Q(\alpha)z(\alpha)\,,\end{aligned}$$

$$\begin{aligned}\frac{d}{d\alpha}\upsilon(\alpha, \theta) &= -\mathrm{Tr}\left\{\Upsilon(\alpha, \theta)G(\alpha)WG^T(\alpha)\right\} \\ &\quad - 2\eta^T(\alpha, \theta)B(\alpha)u_{ext}(\alpha) \\ &\quad - \theta u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) - \theta z^T(\alpha)Q(\alpha)z(\alpha)\,.\end{aligned}$$

At the final time $\alpha = t_f$, it follows that $\varphi(t_f, x(t_f); \theta) = \varrho(t_f, \theta)\exp\{x^T(t_f)\Upsilon(t_f, \theta)x(t_f) + 2x^T(t_f)\eta(t_f, \theta)\} = E\{\exp\{\theta[z(t_f) - y(t_f)]^T Q_f[z(t_f) - y(t_f)]\}\}$ which in turn yields the terminal conditions as

$$\begin{aligned}\Upsilon(t_f, \theta) &= \theta C^T(t_f)Q_f C(t_f)\,, \\ \eta(t_f, \theta) &= -\theta C^T(t_f)Q_f z(t_f)\,, \\ \varrho(t_f, \theta) &= \exp\left\{\theta z^T(t_f)Q_f z(t_f)\right\}\,, \\ \upsilon(t_f, \theta) &= \theta z^T(t_f)Q_f z(t_f)\,.\end{aligned}$$

Remark. The expression for cost cumulants (10) in the tracking problem indicates that additional second and third affine terms are taking into account of dynamics mismatched in their trajectory-governing equations.

By definition, cost cumulants for the tracking problem can be generated by employing the MacLaurin series expansion for the cumulant-generating function

$$\begin{aligned}\psi(\alpha, x_\alpha; \theta) &= \sum_{i=1}^{\infty} \kappa_i(\alpha, x_\alpha) \frac{\theta^i}{i!}, \\ &= \sum_{i=1}^{\infty} \frac{\partial^i}{\partial \theta^i} \psi(\alpha, x_\alpha; \theta)\Big|_{\theta=0} \frac{\theta^i}{i!},\end{aligned} \tag{16}$$

in which $\kappa_i(\alpha, x_\alpha)$ are called cost cumulants. Furthermore, the series coefficients of the expansion is computed by using the result (10)

$$\begin{aligned}\frac{\partial^i}{\partial \theta^i} \psi(\alpha, x_\alpha; \theta)\Big|_{\theta=0} &= x_\alpha^T \frac{\partial^i}{\partial \theta^i} \Upsilon(\alpha, \theta)\Big|_{\theta=0} x_\alpha \\ &+ 2x_\alpha^T \frac{\partial^i}{\partial \theta^i} \eta(\alpha, \theta)\Big|_{\theta=0} + \frac{\partial^i}{\partial \theta^i} \upsilon(\alpha, \theta)\Big|_{\theta=0}.\end{aligned} \tag{17}$$

In view of the results (16) and (17), we may obtain cost cumulants for the tracking problem as described below

$$\begin{aligned}\kappa_i(\alpha, x_\alpha) = x_\alpha^T \frac{\partial^i}{\partial \theta^i} \Upsilon(\alpha, \theta)\Big|_{\theta=0} x_\alpha + 2x_\alpha^T \frac{\partial^i}{\partial \theta^i} \eta(\alpha, \theta)\Big|_{\theta=0} \\ + \frac{\partial^i}{\partial \theta^i} \upsilon(\alpha, \theta)\Big|_{\theta=0},\end{aligned} \tag{18}$$

for any finite $1 \le i < \infty$. For notational convenience, denote $H(\alpha, i) = \frac{\partial^i}{\partial \theta^i} \Upsilon(\alpha, \theta)\Big|_{\theta=0}$, $\breve{D}(\alpha, i) = \frac{\partial^i}{\partial \theta^i} \eta(\alpha, \theta)\Big|_{\theta=0}$, $D(\alpha, i) = \frac{\partial^i}{\partial \theta^i} \upsilon(\alpha, \theta)\Big|_{\theta=0}$. Then, we would like to state the following theorem.

Theorem 2: (Cost Cumulants in Tracking Problems) The system dynamics governed by the linear stochastic differential equations (5)-(6) attempt to track the prescribed signal $z(t)$ with the finite-horizon IQF cost (7). For $k \in \mathbb{Z}^+$ fixed, the kth cost cumulant in the tracking problem is given

$$\begin{aligned}\kappa_k(t_0, x_0; K, u_{ext}) = x_0^T H(t_0, k) x_0 \\ + 2x_0^T \breve{D}(t_0, k) + D(t_0, k),\end{aligned} \tag{19}$$

in which the building variables $\{H(\alpha, i)\}_{i=1}^k$, $\{\breve{D}(\alpha, i)\}_{i=1}^k$, and $\{D(\alpha, i)\}_{i=1}^k$ evaluated at $\alpha = t_0$ satisfy the differential equations (with the dependence of $H(\alpha, i)$, $\breve{D}(\alpha, i)$, and $D(\alpha, i)$ upon u_{ext} and K suppressed)

$$\begin{aligned}\frac{d}{d\alpha} H(\alpha, 1) &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T H(\alpha, 1) \\ &- H(\alpha, 1)\left[A(\alpha) + B(\alpha)K(\alpha)\right] \\ &- C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha),\end{aligned} \tag{20}$$

$$\begin{aligned}\frac{d}{d\alpha} H(\alpha, i) &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T H(\alpha, i) \\ &- H(\alpha, i)\left[A(\alpha) + B(\alpha)K(\alpha)\right] \\ &- \sum_{j=1}^{i-1} \frac{2i!}{j!(i-j)!} H(\alpha, j) G(\alpha) W G^T(\alpha) H(\alpha, i-j)\end{aligned} \tag{21}$$

together with

$$\begin{aligned}\frac{d}{d\alpha} \breve{D}(\alpha, 1) &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \breve{D}(\alpha, 1) \\ &- H(\alpha, 1)B(\alpha)u_{ext}(\alpha) \\ &- K^T(\alpha)R(\alpha)u_{ext}(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha),\end{aligned} \tag{22}$$

$$\begin{aligned}\frac{d}{d\alpha} \breve{D}(\alpha, i) &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \breve{D}(\alpha, i) \\ &- H(\alpha, i)B(\alpha)u_{ext}(\alpha), \quad 2 \le i \le k,\end{aligned} \tag{23}$$

and

$$\begin{aligned}\frac{d}{d\alpha} D(\alpha, 1) &= -\mathrm{Tr}\left\{H(\alpha, 1)G(\alpha)WG^T(\alpha)\right\} \\ &- 2\breve{D}^T(\alpha, 1)B(\alpha)u_{ext}(\alpha) \\ &- u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha),\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{d}{d\alpha} D(\alpha, i) &= -\mathrm{Tr}\left\{H(\alpha, i)G(\alpha)WG^T(\alpha)\right\} \\ &- 2\breve{D}^T(\alpha, i)B(\alpha)u_{ext}(\alpha), \ 2 \le i \le k\end{aligned} \tag{25}$$

where the terminal conditions $H(t_f, 1) = C^T(t_f)Q_f C(t_f)$, $H(t_f, i) = 0$ for $2 \le i \le k$; $\breve{D}(t_f, 1) = -C^T(t_f)Q_f z(t_f)$, $\breve{D}(t_f, i) = 0$ for $2 \le i \le k$ and $D(t_f, 1) = z^T(t_f)Q_f z(t_f)$, $D(t_f, i) = 0$ for $2 \le i \le k$.

Proof. Note that the equations (20), (22) and (24) satisfied by $H(\alpha, 1)$, $\breve{D}(\alpha, 1)$, and $D(\alpha, 1)$ can be obtained by taking the derivative with respect to θ of the equations (11)-(13)

$$\begin{aligned}\frac{d}{d\alpha}\left\{\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\right\} &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta) \\ &-\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\left[A(\alpha) + B(\alpha)K(\alpha)\right] \\ &-2\left\{\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\right\}G(\alpha)WG^T(\alpha)\Upsilon(\alpha,\theta) \\ &-2\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\left\{\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\right\} \\ &-C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha),\end{aligned}$$

$$\begin{aligned}\frac{d}{d\alpha}\left\{\frac{\partial}{\partial\theta}\eta(\alpha,\theta)\right\} &= -\left[A(\alpha) + B(\alpha)K(\alpha)\right]^T \frac{\partial}{\partial\theta}\eta(\alpha,\theta) \\ &-\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)B(\alpha)u_{ext}(\alpha) \\ &-K^T(\alpha)R(\alpha)u_{ext}(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha),\end{aligned}$$

$$\begin{aligned}\frac{d}{d\alpha}\left\{\frac{\partial}{\partial\theta}\upsilon(\alpha,\theta)\right\} &= -\mathrm{Tr}\left\{\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\} \\ &-2\frac{\partial}{\partial\theta}\eta^T(\alpha,\theta)B(\alpha)u_{ext}(\alpha) \\ &-u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha),\end{aligned}$$

wherein terminal conditions $\frac{\partial}{\partial\theta}\Upsilon(t_f, \theta) = C^T(t_f)Q_f C(t_f)$, $\frac{\partial}{\partial\theta}\eta(t_f, \theta) = -C^T(t_f)Q_f z(t_f)$, and $\frac{\partial}{\partial\theta}\upsilon(t_f, \theta) = z^T(t_f)Q_f z(t_f)$. It is important to see that when $\theta = 0$ the equation (11) becomes

$$\begin{aligned}\frac{d}{d\alpha}\Upsilon(\alpha, 0) &= -[A(\alpha) + B(\alpha)K(\alpha)]^T\Upsilon(\alpha, 0) \\ &- \Upsilon(\alpha, 0)[A(\alpha) + B(\alpha)K(\alpha)] \\ &- 2\Upsilon(\alpha, 0)G(\alpha)WG^T(\alpha)\Upsilon(\alpha, 0), \qquad \Upsilon(t_f, 0) = 0.\end{aligned}$$

Because the closed-loop matrix $A(\alpha)+B(\alpha)K(\alpha)$ is assumed stable for all $\alpha \in [t_0,t_f]$, it is then deduced that $\Upsilon(\alpha,0)=0$. Using this result together with the definitions of $H(\alpha,1) = \frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\big|_{\theta=0}$, $\breve{D}(\alpha,1) = \frac{\partial}{\partial\theta}\eta(\alpha,\theta)\big|_{\theta=0}$, and $D(\alpha,1) = \frac{\partial}{\partial\theta}\upsilon(\alpha,\theta)\big|_{\theta=0}$, the first cost cumulant is found

$$\kappa_1(t_0,x_0;K) = x_0^T H(t_0,1)x_0 + 2x_0^T \breve{D}(t_0,1) + D(t_0,1)\,,$$

where the solutions $H(\alpha,1)$, $\breve{D}(\alpha,1)$ and $D(\alpha,1)$ satisfy the backward-in-time differential equations

$$\begin{aligned}
\frac{d}{d\alpha}H(\alpha,1) &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T H(\alpha,1)\\
&\quad - H(\alpha,1)\left[A(\alpha)+B(\alpha)K(\alpha)\right]\\
&\quad - C^T(\alpha)Q(\alpha)C(\alpha) - K^T(\alpha)R(\alpha)K(\alpha)\,,\\
\frac{d}{d\alpha}\breve{D}(\alpha,1) &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{D}(\alpha,1)\\
&\quad - H(\alpha,1)B(\alpha)u_{ext}(\alpha)\\
&\quad - K^T(\alpha)R(\alpha)u_{ext}(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha)\,,\\
\frac{d}{d\alpha}D(\alpha,1) &= -\mathrm{Tr}\left\{H(\alpha,1)G(\alpha)WG^T(\alpha)\right\}\\
&\quad - 2\breve{D}^T(\alpha,1)B(\alpha)u_{ext}(\alpha)\\
&\quad - u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha) - z^T(\alpha)Q(\alpha)z(\alpha)\,,
\end{aligned}$$

with boundaries $H(t_f,1) = C^T(t_f)Q_fC(t_f)$, $\breve{D}(t_f,1) = -C^T(t_f)Q_fz(t_f)$ and $D(t_f,1) = z^T(t_f)Q_fz(t_f)$. Repeatedly, taking $\frac{\partial^2}{\partial\theta^2}$ of the equations (11)-(13) yield the corresponding differential equations

$$\begin{aligned}
\frac{d}{d\alpha}\left\{\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)\right\} &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)\\
&\quad -\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)\left[A(\alpha)+B(\alpha)K(\alpha)\right]\\
&\quad -2\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\Upsilon(\alpha,\theta)\\
&\quad -4\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\\
&\quad -2\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)\,,\\
\frac{d}{d\alpha}\left\{\frac{\partial^2}{\partial\theta^2}\eta(\alpha,\theta)\right\} &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\frac{\partial^2}{\partial\theta^2}\eta(\alpha,\theta)\\
&\quad -\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)B(\alpha)u_{ext}(\alpha)\,,\\
\frac{d}{d\alpha}\left\{\frac{\partial^2}{\partial\theta^2}\upsilon(\alpha,\theta)\right\} &= -\mathrm{Tr}\left\{\frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)G(\alpha)WG^T(\alpha)\right\}\\
&\quad -2\frac{\partial^2}{\partial\theta^2}\eta^T(\alpha,\theta)B(\alpha)u_{ext}(\alpha)\,,
\end{aligned}$$

together with terminal conditions $\frac{\partial^2}{\partial\theta^2}\Upsilon(t_f,\theta) = 0$, $\frac{\partial^2}{\partial\theta^2}\eta(t_f,\theta) = 0$, and $\frac{\partial^2}{\partial\theta^2}\upsilon(t_f,\theta) = 0$. Having substituted $H(\alpha,1) = \frac{\partial}{\partial\theta}\Upsilon(\alpha,\theta)\big|_{\theta=0}$, $H(\alpha,2) = \frac{\partial^2}{\partial\theta^2}\Upsilon(\alpha,\theta)\Big|_{\theta=0}$, $\breve{D}(\alpha,2) = \frac{\partial^2}{\partial\theta^2}\eta(\alpha,\theta)\Big|_{\theta=0}$, $D(\alpha,2) = \frac{\partial^2}{\partial\theta^2}\upsilon(\alpha,\theta)\Big|_{\theta=0}$ and $\Upsilon(\alpha,\theta)|_{\theta=0} = 0$ into the above equations, the second cost cumulant can be obtained as follows

$$\kappa_2(t_0,x_0;K) = x_0^T H(t_0,2)x_0 + 2x_0^T \breve{D}(t_0,2) + D(t_0,2)\,,$$

in which the solutions $H(\alpha,2)$, $\breve{D}(\alpha,2)$ and $D(\alpha,2)$ evaluated at $\alpha=t_0$, are solving the differential equations

$$\begin{aligned}
\frac{d}{d\alpha}H(\alpha,2) &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T H(\alpha,2)\\
&\quad -H(\alpha,2)\left[A(\alpha)+B(\alpha)K(\alpha)\right]\\
&\quad -4H(\alpha,1)G(\alpha)WG^T(\alpha)H(\alpha,1)\,, \qquad H(t_f,2)=0\,,\\
\frac{d}{d\alpha}\breve{D}(\alpha,2) &= -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{D}(\alpha,2)\\
&\quad -H(\alpha,2)B(\alpha)u_{ext}(\alpha)\,, \qquad \breve{D}(t_f,2)=0\,,\\
\frac{d}{d\alpha}D(\alpha,2) &= -\mathrm{Tr}\left\{H(\alpha,2)G(\alpha)WG^T(\alpha)\right\}\\
&\quad -2\breve{D}^T(\alpha,2)B(\alpha)u_{ext}(\alpha)\,, \qquad D(t_f,2)=0\,.
\end{aligned}$$

By successively taking derivatives of the equations (11)-(13) with respect to θ and evaluating the results at $\theta=0$, the ith cost cumulant can be written for all $2\le i\le k$

$$\kappa_i(t_0,x_0;K) = x_0^T H(t_0,i)x_0 + 2x_0^T \breve{D}(t_0,i) + D(t_0,i)\,,$$

where $H(\alpha,i)$, $\breve{D}(\alpha,i)$ and $D(\alpha,i)$ evaluated at $\alpha=t_0$ are the solutions of the coupled differential equations

$$\begin{aligned}
&\frac{d}{d\alpha}H(\alpha,i) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T H(\alpha,i)\\
&- H(\alpha,i)\left[A(\alpha)+B(\alpha)K(\alpha)\right]\\
&-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}H(\alpha,j)G(\alpha)WG^T(\alpha)H(\alpha,i-j), H(t_f,i)=0\\
&\frac{d}{d\alpha}\breve{D}(\alpha,i) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T \breve{D}(\alpha,i)\\
&- H(\alpha,i)B(\alpha)u_{ext}(\alpha)\,, \quad \breve{D}(t_f,i)=0\,,\\
&\frac{d}{d\alpha}D(\alpha,i) = -\mathrm{Tr}\left\{H(\alpha,i)G(\alpha)WG^T(\alpha)\right\}\\
&- 2\breve{D}^T(\alpha,i)B(\alpha)u_{ext}(\alpha)\,, \quad D(t_f,i)=0\,.
\end{aligned}$$

Thus, the proof is now complete.

II. Problem Statements

In preparing for the kCC control statements of the tracking problem, let k-tuple variables $\mathcal{H}$ and $\mathcal{D}$ be defined as follows $\mathcal{H}(\cdot) = (\mathcal{H}_1(\cdot),\ldots,\mathcal{H}_k(\cdot))$, $\breve{\mathcal{D}}(\cdot) = \left(\breve{\mathcal{D}}_1(\cdot),\ldots,\breve{\mathcal{D}}_k(\cdot)\right)$, $\mathcal{D}(\cdot) = (\mathcal{D}_1(\cdot),\ldots,\mathcal{D}_k(\cdot))$ for each element $\mathcal{H}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n\times n})$ of $\mathcal{H}$, $\breve{\mathcal{D}}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^{n})$ of $\breve{\mathcal{D}}$ and $\mathcal{D}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ of $\mathcal{D}$ having the representations

$$\mathcal{H}_i(\cdot) = H(\cdot,i)\,, \qquad \breve{\mathcal{D}}_i(\cdot) = \breve{D}(\cdot,i)\,, \qquad \mathcal{D}_i(\cdot) = D(\cdot,i)\,,$$

with the right members satisfying the dynamic equations (20)-(25) on the horizon $[t_0,t_f]$. The problem formulation can be considerably simplified if the convenient mappings are introduced

$$\begin{aligned}
&\mathcal{F}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times\mathbb{R}^{m\times n} \mapsto \mathbb{R}^{n\times n}\\
&\breve{\mathcal{G}}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^{n})^k\times\mathbb{R}^{m\times n}\times\mathbb{R}^{m} \mapsto \mathbb{R}^{n}\\
&\mathcal{G}_i : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^{n})^k\times\mathbb{R}^{m} \mapsto \mathbb{R}
\end{aligned}$$

where the actions are given by

$$\mathcal{F}_1(\alpha,\mathcal{H},K) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\mathcal{H}_1(\alpha)$$
$$-\mathcal{H}_1(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right]$$
$$-C^T(\alpha)Q(\alpha)C(\alpha)-K^T(\alpha)R(\alpha)K(\alpha)\,,$$
$$\mathcal{F}_i(\alpha,\mathcal{H},K) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\mathcal{H}_i(\alpha)$$
$$-\mathcal{H}_i(\alpha)\left[A(\alpha)+B(\alpha)K(\alpha)\right]$$
$$-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}_{i-j}(\alpha)\,,$$
$$\breve{\mathcal{G}}_1\left(\alpha,\mathcal{H},\breve{\mathcal{D}},K,u_{ext}\right) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\breve{\mathcal{D}}_1(\alpha)$$
$$-\mathcal{H}_1(\alpha)B(\alpha)u_{ext}(\alpha)$$
$$-K^T(\alpha)R(\alpha)u_{ext}(\alpha)+C^T(\alpha)Q(\alpha)z(\alpha)\,,$$
$$\breve{\mathcal{G}}_i\left(\alpha,\mathcal{H},\breve{\mathcal{D}},K,u_{ext}\right) = -\left[A(\alpha)+B(\alpha)K(\alpha)\right]^T\breve{\mathcal{D}}_i(\alpha)$$
$$-\mathcal{H}_i(\alpha)B(\alpha)u_{ext}(\alpha)\,,$$
$$\mathcal{G}_1\left(\alpha,\mathcal{H},\breve{\mathcal{D}},u_{ext}\right) = -\mathrm{Tr}\left\{\mathcal{H}_1(\alpha)G(\alpha)WG^T(\alpha)\right\}$$
$$-2\breve{\mathcal{D}}_1^T(\alpha)B(\alpha)u_{ext}(\alpha)$$
$$-u_{ext}^T(\alpha)R(\alpha)u_{ext}(\alpha)-z^T(\alpha)Q(\alpha)z(\alpha)\,,$$
$$\mathcal{G}_i\left(\alpha,\mathcal{H},\breve{\mathcal{D}},u_{ext}\right) = -\mathrm{Tr}\left\{\mathcal{H}_i(\alpha)G(\alpha)WG^T(\alpha)\right\}$$
$$-2\breve{\mathcal{D}}_i^T(\alpha)B(\alpha)u_{ext}(\alpha)\,.$$

Now there is no difficulty to establish the product mappings

$$\mathcal{F}_1\times\cdots\times\mathcal{F}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times\mathbb{R}^{m\times n}\mapsto(\mathbb{R}^{n\times n})^k$$
$$\breve{\mathcal{G}}_1\times\cdots\times\breve{\mathcal{G}}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^n)^k\times\mathbb{R}^{m\times n}\times\mathbb{R}^m\mapsto(\mathbb{R}^n)^k$$
$$\mathcal{G}_1\times\cdots\times\mathcal{G}_k : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^n)^k\times\mathbb{R}^m\mapsto\mathbb{R}^k$$

along with the corresponding notations $\mathcal{F}=\mathcal{F}_1\times\cdots\times\mathcal{F}_k$, $\breve{\mathcal{G}}=\breve{\mathcal{G}}_1\times\cdots\times\breve{\mathcal{G}}_k$, and $\mathcal{G}=\mathcal{G}_1\times\cdots\times\mathcal{G}_k$. Thus, the dynamic equations of motion (20)-(25) can be rewritten as

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha)),\ \mathcal{H}(t_f)=\mathcal{H}_f,$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}(\alpha)=\breve{\mathcal{G}}\Big(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),K(\alpha),u_{ext}(\alpha)\Big),\ \breve{\mathcal{D}}(t_f)=\breve{\mathcal{D}}_f,$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}\left(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),u_{ext}(\alpha)\right),\ \mathcal{D}(t_f)=\mathcal{D}_f$$

where k-tuple values $\mathcal{H}_f = \left(C^T(t_f)Q_fC(t_f),0,\ldots,0\right)$, $\breve{\mathcal{D}}_f=\left(-C^T(t_f)Q_fz(t_f),0,\ldots,0\right)$ and $\mathcal{D}_f=(0,\ldots,0)$.

Note that the product system uniquely determines $\mathcal{H}$, $\breve{\mathcal{D}}$ and $\mathcal{D}$ once the admissible external signal u_{ext} and feedback gain K are specified. Hence, they are considered as $\mathcal{H}=\mathcal{H}(\cdot,K)$, $\breve{\mathcal{D}}=\breve{\mathcal{D}}(\cdot,K,u_{ext})$ and $\mathcal{D}=\mathcal{D}(\cdot,K,u_{ext})$. The performance index in kCC control problems can now be formulated in u_{ext} and K.

Definition 1: (Performance Index)
Fix $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\geq 0\}_{i=1}^k$ with $\mu_1>0$. Then for the given (t_0,x_0), the performance index

$$\phi_{tk} : [t_0,t_f]\times(\mathbb{R}^{n\times n})^k\times(\mathbb{R}^n)^k\times\mathbb{R}^k\mapsto\mathbb{R}^+$$

in finite-horizon state-feedback kCC control for the tracking problem is defined as

$$\phi_{tk}\left(t_0,\mathcal{H}(t_0,K),\breve{\mathcal{D}}(t_0,K,u_{ext}),\mathcal{D}(t_0,K,u_{ext})\right)$$
$$=\sum_{i=1}^{k}\mu_i\big[x_0^T\mathcal{H}_i(t_0,K)x_0$$
$$+2x_0^T\breve{\mathcal{D}}_i(t_0,K,u_{ext})+\mathcal{D}_i(t_0,K,u_{ext})\big] \tag{26}$$

where the scalar, real constants μ_i represent parametric design freedom and the unique solutions $\{\mathcal{H}_i(t_0,K)\geq 0\}_{i=1}^k$, $\left\{\breve{\mathcal{D}}_i(t_0,K,u_{ext})\right\}_{i=1}^k$ and $\{\mathcal{D}_i(t_0,K,u_{ext})\}_{i=1}^k$ evaluated at $\alpha=t_0$ satisfy the dynamic equations

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha)),\ \mathcal{H}(t_f)=\mathcal{H}_f,$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}(\alpha)=\breve{\mathcal{G}}\Big(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),K(\alpha),u_{ext}(\alpha)\Big),\ \breve{\mathcal{D}}(t_f)=\breve{\mathcal{D}}_f,$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}\left(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),u_{ext}(\alpha)\right),\ \mathcal{D}(t_f)=\mathcal{D}_f\,.$$

For given terminal data $(t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f)$, the classes of admissible external signal and feedback gains may be defined as follows.

Definition 2: (Admissible Signal and Feedback Gains)
Let compact subsets $\overline{U}\subset\mathbb{R}^m$ and $\overline{K}\subset\mathbb{R}^{m\times n}$ be the sets of allowable external inputs and gain values. For the given $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\geq 0\}_{i=1}^k$ with $\mu_1>0$, the set of admissible external signals $\mathcal{U}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ and feedback gains $\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ are respectively assumed to be the classes of $\mathcal{C}([t_0,t_f];\mathbb{R}^m)$ and $\mathcal{C}([t_0,t_f];\mathbb{R}^{m\times n})$ with values $u_{ext}(\cdot)\in\overline{U}$ and $K(\cdot)\in\overline{K}$ for which solutions to the dynamic equations with the terminal conditions $\mathcal{H}(t_f)=\mathcal{H}_f$, $\breve{\mathcal{D}}(t_f)=\breve{\mathcal{D}}_f$, and $\mathcal{D}(t_f)=\mathcal{D}_f$

$$\frac{d}{d\alpha}\mathcal{H}(\alpha)=\mathcal{F}(\alpha,\mathcal{H}(\alpha),K(\alpha))\,, \tag{27}$$
$$\frac{d}{d\alpha}\breve{\mathcal{D}}(\alpha)=\breve{\mathcal{G}}\left(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),K(\alpha),u_{ext}(\alpha)\right)\,, \tag{28}$$
$$\frac{d}{d\alpha}\mathcal{D}(\alpha)=\mathcal{G}\left(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),u_{ext}(\alpha)\right) \tag{29}$$

exist on the interval of optimization $[t_0,t_f]$.
Then the optimization statements for the state-feedback kCC control of the tracking problem over a finite horizon may be stated in the sequel.

Definition 3: (Optimization Problem)
Suppose that $k\in\mathbb{Z}^+$ and the sequence $\mu=\{\mu_i\geq 0\}_{i=1}^k$ with $\mu_1>0$ are fixed. Then the state-feedback kCC control optimization problem over $[t_0,t_f]$ is given by the minimization of (26) over $u_{ext}(\cdot)\in\mathcal{U}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$, $K(\cdot)\in\mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ and subject to the dynamic equations of motion (27)-(29) for $\alpha\in[t_0,t_f]$.
The sequence of following results will discuss the construction of scalar-valued functions which are the candidates for the value function.

Definition 4: (Reachable Set)
Let reachable set $\mathcal{Q}$ be defined $\mathcal{Q}\triangleq\left\{\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)\in\right.$

$[t_0, t_f] \times (\mathbb{R}^{n\times n})^k \times (\mathbb{R}^n)^k \times \mathbb{R}^k\Big\}$ such that $\mathcal{U}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu} \neq 0$ and $\mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu} \neq 0$.

By adapting to the initial cost problem and the terminologies present in the kCC control, the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the value function $\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ is then given as follows.

Theorem 3: (HJB Equation-Mayer Problem)
Let $\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ be any interior point of the reachable set $\mathcal{Q}$ at which the value function $\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ is differentiable. If there exist optimal external signal $u^*_{ext} \in \mathcal{U}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$ and feedback gain $K^* \in \mathcal{K}_{\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z};\mu}$, then the partial differential equation of dynamic programming

$$
\begin{aligned}
0 = \min_{u_{ext}\in\overline{U},\, K\in\overline{K}} \Bigg\{ & \frac{\partial}{\partial\varepsilon}\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right) \\
&+ \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Y})}\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)\mathrm{vec}\left(\mathcal{F}\left(\varepsilon,\mathcal{Y},K\right)\right) \\
&+ \frac{\partial}{\partial\,\mathrm{vec}\left(\breve{\mathcal{Z}}\right)}\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)\mathrm{vec}\left(\breve{\mathcal{G}}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,u_{ext}\right)\right) \\
&+ \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)\mathrm{vec}\left(\mathcal{G}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},u_{ext}\right)\right)\Bigg\}
\end{aligned}
\tag{30}
$$

is satisfied wherein the boundary condition $\mathcal{V}\left(t_0,\mathcal{H}_0,\breve{\mathcal{D}}_0,\mathcal{D}_0\right) = \phi_{tk}\left(t_0,\mathcal{H}_0,\breve{\mathcal{D}}_0,\mathcal{D}_0\right)$.

Proof. Refer to the reference [7] for the detailed proof.

Theorem 4: (Verification Theorem)
Fix $k \in \mathbb{Z}^+$ and let $\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ be a continuously differentiable solution of the HJB equation (30) which satisfies the boundary condition

$$
\mathcal{W}\left(t_0,\mathcal{H}_0,\breve{\mathcal{D}}_0,\mathcal{D}_0\right) = \phi_{tk}\left(t_0,\mathcal{H}_0,\breve{\mathcal{D}}_0,\mathcal{D}_0\right). \tag{31}
$$

Let $(t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f)$ be in $\mathcal{Q}$; (u_{ext},K) in $\mathcal{U}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu} \times \mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$; $\mathcal{H}$, $\breve{\mathcal{D}}$ and $\mathcal{D}$ the corresponding solutions of (27)-(29). Then $\mathcal{W}(\alpha,\mathcal{H}(\alpha),\breve{\mathcal{D}}(\alpha),\mathcal{D}(\alpha))$ is a non-increasing function of α. If (u^*_{ext},K^*) is in $\mathcal{U}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu} \times \mathcal{K}_{t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f;\mu}$ defined on $[t_0,t_f]$ with corresponding solutions, $\mathcal{H}^*$, $\breve{\mathcal{D}}^*$, and $\mathcal{D}^*$ of (27)-(29) such that for $\alpha \in [t_0,t_f]$

$$
\begin{aligned}
0 = &\frac{\partial}{\partial\varepsilon}\mathcal{W}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),\mathcal{D}^*(\alpha)\right) \\
&+ \frac{\partial}{\partial\mathrm{vec}(\mathcal{Y})}\mathcal{W}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),\mathcal{D}^*(\alpha)\right)\cdot \\
&\quad\cdot\mathrm{vec}\left(\mathcal{F}\left(\alpha,\mathcal{H}^*(\alpha),K^*(\alpha)\right)\right) \\
&+ \frac{\partial}{\partial\mathrm{vec}(\breve{\mathcal{Z}})}\mathcal{W}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),\mathcal{D}^*(\alpha)\right)\cdot \\
&\quad\cdot\mathrm{vec}\left(\breve{\mathcal{G}}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),K^*(\alpha),u^*_{ext}(\alpha)\right)\right) \\
&+ \frac{\partial}{\partial\,\mathrm{vec}(\mathcal{Z})}\mathcal{W}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),\mathcal{D}^*(\alpha)\right)\cdot \\
&\quad\cdot\mathrm{vec}\left(\mathcal{G}\left(\alpha,\mathcal{H}^*(\alpha),\breve{\mathcal{D}}^*(\alpha),u^*_{ext}(\alpha)\right)\right),
\end{aligned}
\tag{32}
$$

then u^*_{ext} and K^* are optimal. Moreover,

$$
\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right) = \mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right) \tag{33}
$$

where $\mathcal{V}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ is the value function.

Proof. Refer to [7].

III. Optimal Solution of kCC Control

The treatment of HJB approach to obtaining a state-feedback solution to the kCC control problem over the finite horizon of optimization requires to parameterize the terminal time and states of the dynamical equations as $\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ rather than $\left(t_f,\mathcal{H}_f,\breve{\mathcal{D}}_f,\mathcal{D}_f\right)$. That is, for $\varepsilon \in [t_0,t_f]$ and $1 \le i \le k$, the states of the system (27)-(29) defined on the interval $[t_0,\varepsilon]$ have the terminal values denoted by

$$
\mathcal{H}(\varepsilon) = \mathcal{Y}, \qquad \breve{\mathcal{D}}(\varepsilon) = \breve{\mathcal{Z}}, \qquad \mathcal{D}(\varepsilon) = \mathcal{Z}.
$$

Observe that the performance index (26) is quadratic affine in terms of the arbitrarily fixed x_0. This suggests a solution to the HJB equation (30) may be sought in the form

$$
\begin{aligned}
\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right) &= x_0^T\sum_{i=1}^{k}\mu_i\left(\mathcal{Y}_i+\mathcal{E}_i(\varepsilon)\right)x_0 \\
&+ 2x_0^T\sum_{i=1}^{k}\mu_i\left(\breve{\mathcal{Z}}_i+\breve{\mathcal{T}}_i(\varepsilon)\right) + \sum_{i=1}^{k}\mu_i\left(\mathcal{Z}_i+\mathcal{T}_i(\varepsilon)\right)
\end{aligned}
\tag{34}
$$

where these parametric functions of time $\mathcal{E}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{S}^n)$, $\breve{\mathcal{T}}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R}^n)$ and $\mathcal{T}_i \in \mathcal{C}^1([t_0,t_f];\mathbb{R})$ are to be determined. Using the isomorphic vec mapping, there is no difficulty to verify the following result.

Lemma 1: Fix $k \in \mathbb{Z}^+$ and let $\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ be any interior point of the reachable set $\mathcal{Q}$ at which the real-valued function $\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ of the form (34) is differentiable. The derivative of $\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right)$ with respect to ε is given

$$
\begin{aligned}
\frac{d}{d\varepsilon}\mathcal{W}\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},\mathcal{Z}\right) &= x_0^T\sum_{i=1}^{k}\mu_i\left(\mathcal{F}_i(\varepsilon,\mathcal{Y},K)+\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)\right)x_0 \\
&+ 2x_0^T\sum_{i=1}^{k}\mu_i\left(\breve{\mathcal{G}}_i\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,u_{ext}\right)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon)\right) \\
&+ \sum_{i=1}^{k}\mu_i\left(\mathcal{G}_i\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},u_{ext}\right)+\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon)\right),
\end{aligned}
\tag{35}
$$

provided $u_{ext} \in \overline{U}$ and $K \in \overline{K}$.

Trying the guess solution (34) into the HJB equation (30), it follows that

$$
\begin{aligned}
0 = \min_{u_{ext}\in\overline{U},\, K\in\overline{K}} \Bigg\{ & x_0^T\sum_{i=1}^{k}\mu_i\left(\mathcal{F}_i(\varepsilon,\mathcal{Y},K)+\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)\right)x_0 \\
&+ 2x_0^T\sum_{i=1}^{k}\mu_i\left(\breve{\mathcal{G}}_i\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,u_{ext}\right)+\frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon)\right) \\
&+ \sum_{i=1}^{k}\mu_i\left(\mathcal{G}_i\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},u_{ext}\right)+\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon)\right)\Bigg\}.
\end{aligned}
\tag{36}
$$

Notice that

$$\sum_{i=1}^{k}\mu_i\mathcal{F}_i(\varepsilon,\mathcal{Y},K)=-\left[A(\varepsilon)+B(\varepsilon)K\right]^T\sum_{i=1}^{k}\mu_i\mathcal{Y}_i-\sum_{i=1}^{k}\mu_i\mathcal{Y}_i\left[A(\varepsilon)+B(\varepsilon)K\right]-\mu_1C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)$$
$$-\mu_1K^TR(\varepsilon)K-\sum_{i=2}^{k}\mu_i\sum_{j=1}^{i-1}\frac{2!}{j!(i-j)!}\mathcal{Y}_jG(\varepsilon)WG^T(\varepsilon)\mathcal{Y}_{i-j},$$

$$\sum_{i=1}^{k}\mu_i\breve{\mathcal{G}}_i\Big(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},K,u_{ext}\Big)=-\left[A(\varepsilon)+B(\varepsilon)K\right]^T\sum_{i=1}^{k}\mu_i\breve{\mathcal{Z}}_i$$
$$-\sum_{i=1}^{k}\mu_i\mathcal{Y}_iB(\varepsilon)u_{ext}-\mu_1K^TR(\varepsilon)u_{ext}+\mu_1C^T(\varepsilon)Q(\varepsilon)z(\varepsilon),$$

$$\sum_{i=1}^{k}\mu_i\mathcal{G}_i\left(\varepsilon,\mathcal{Y},\breve{\mathcal{Z}},u_{ext}\right)=-\sum_{i=1}^{k}\mu_i\mathrm{Tr}\left\{\mathcal{Y}_iG(\varepsilon)WG^T(\varepsilon)\right\}$$
$$-2\sum_{i=1}^{k}\mu_i\breve{\mathcal{Z}}_i^TB(\varepsilon)u_{ext}-\mu_1u_{ext}^TR(\varepsilon)u_{ext}-\mu_1z^T(\varepsilon)Q(\varepsilon)z(\varepsilon).$$

Since the initial condition x_0 and M_0 are arbitrary vector and rank-one matrix, the necessary condition for an extremum of (26) on $[t_0,\varepsilon]$ is obtained by differentiating the expression within the bracket of (36) with respect to u_{ext} and K as

$$u_{ext}(\varepsilon,\breve{\mathcal{Z}})=-R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{Z}}_r\,,\tag{37}$$

$$K(\varepsilon,\mathcal{Y})=-R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_r\,,\tag{38}$$

where $\widehat{\mu}_r=\mu_i/\mu_1$ and $\mu_1>0$. Replacing (37) and (38) into (36) leads to the value function

$$x_0^T\Bigg[\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)-A^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i-\sum_{i=1}^{k}\mu_i\mathcal{Y}_iA(\varepsilon)-\mu_1C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)$$
$$+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_rB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{i=1}^{k}\mu_i\mathcal{Y}_i+\sum_{i=1}^{k}\mu_i\mathcal{Y}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s$$
$$-\mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_rB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{Y}_s-\sum_{i=2}^{k}\mu_i\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{Y}_jG(\varepsilon)WG^T(\varepsilon)\mathcal{Y}_{i-j}\Bigg]x_0$$
$$+2x_0^T\Bigg[\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon)-A^T(\varepsilon)\sum_{i=1}^{k}\mu_i\breve{\mathcal{Z}}_i+\mu_1C^T(\varepsilon)Q(\varepsilon)z(\varepsilon)+\sum_{r=1}^{k}\mu_r\mathcal{Y}_rB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{i=1}^{k}\mu_i\breve{\mathcal{Z}}_i$$
$$+\sum_{i=1}^{k}\mu_i\mathcal{Y}_iB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{Z}}_r-\mu_1\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{Y}_rB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{Z}}_s\Bigg]$$
$$+\sum_{i=1}^{k}\mu_i\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon)-\sum_{i=1}^{k}\mu_i\mathrm{Tr}\left\{\mathcal{Y}_iG(\varepsilon)WG^T(\varepsilon)\right\}$$
$$+2\sum_{i=1}^{k}\mu_i\breve{\mathcal{Z}}_i^TB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{Z}}_r-\mu_1z^T(\varepsilon)Q(\varepsilon)z(\varepsilon)$$
$$-\mu_1\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{Z}}_r^TB(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{Z}}_s\,.\tag{39}$$

The remaining task is to display time-dependent functions $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\left\{\breve{\mathcal{T}}_i(\cdot)\right\}_{i=1}^k$, and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$, which yield a sufficient condition to have the left-hand side of (39) being zero for any $\varepsilon\in[t_0,t_f]$, when $\{\mathcal{Y}_i\}_{i=1}^k$ and $\left\{\breve{\mathcal{Z}}_i\right\}_{i=1}^k$ are evaluated along solutions to the cumulant-generating equations. Careful observation of (39) suggests that $\{\mathcal{E}_i(\cdot)\}_{i=1}^k$, $\left\{\breve{\mathcal{T}}_i(\cdot)\right\}_{i=1}^k$ and $\{\mathcal{T}_i(\cdot)\}_{i=1}^k$ may be chosen to satisfy the differential equations as follows

$$\frac{d}{d\varepsilon}\mathcal{E}_1(\varepsilon)=A^T(\varepsilon)\mathcal{H}_1(\varepsilon)+\mathcal{H}_1(\varepsilon)A(\varepsilon)+C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)-\mathcal{H}_1(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_1(\varepsilon)+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\,,\tag{40}$$

$$\frac{d}{d\varepsilon}\mathcal{E}_i(\varepsilon)=A^T(\varepsilon)\mathcal{H}_i(\varepsilon)+\mathcal{H}_i(\varepsilon)A(\varepsilon)-\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_i(\varepsilon)+\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\,,\tag{41}$$

$$\frac{d}{d\varepsilon}\breve{\mathcal{T}}_1(\varepsilon)=A^T(\varepsilon)\breve{\mathcal{D}}_1(\varepsilon)-C^T(\varepsilon)Q(\varepsilon)z(\varepsilon)-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\breve{\mathcal{D}}_1(\varepsilon)$$
$$-\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{D}}_s(\varepsilon)\,,\tag{42}$$

$$\frac{d}{d\varepsilon}\breve{\mathcal{T}}_i(\varepsilon) = A^T(\varepsilon)\breve{\mathcal{D}}_i(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\breve{\mathcal{D}}_i(\varepsilon)$$
$$-\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)\,, \tag{43}$$
$$\frac{d}{d\varepsilon}\mathcal{T}_1(\varepsilon) = \mathrm{Tr}\left\{\mathcal{H}_1(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\} + z^T(\varepsilon)Q(\varepsilon)z(\varepsilon)$$
$$-2\breve{\mathcal{D}}_1^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)$$
$$+\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{D}}_s(\varepsilon)\,, \tag{44}$$
$$\frac{d}{d\varepsilon}\mathcal{T}_i(\varepsilon) = \mathrm{Tr}\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}$$
$$-2\breve{\mathcal{D}}_i^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)\,. \tag{45}$$

The external signal and feedback gain specified in (37) and (38) are now applied along the solution trajectories of the equations (27)-(29)

$$\frac{d}{d\varepsilon}\mathcal{H}_1(\varepsilon) = -A^T(\varepsilon)\mathcal{H}_1(\varepsilon) - \mathcal{H}_1(\varepsilon)A(\varepsilon) - C^T(\varepsilon)Q(\varepsilon)C(\varepsilon)$$
$$+\mathcal{H}_1(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$
$$+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_1(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)\,, \tag{46}$$
$$\frac{d}{d\varepsilon}\mathcal{H}_i(\varepsilon) = -A^T(\varepsilon)\mathcal{H}_i(\varepsilon) - \mathcal{H}_i(\varepsilon)A(\varepsilon)$$
$$+\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\mathcal{H}_s(\varepsilon)$$
$$+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\mathcal{H}_i(\varepsilon)$$
$$-\sum_{j=1}^{i-1}\frac{2i!}{j!(i-j)!}\mathcal{H}_j(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\mathcal{H}_{i-j}(\varepsilon)\,, \tag{47}$$
$$\frac{d}{d\varepsilon}\breve{\mathcal{D}}_1(\varepsilon) = -A^T(\varepsilon)\breve{\mathcal{D}}_1(\varepsilon) + C^T(\varepsilon)Q(\varepsilon)z(\varepsilon)$$
$$+\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\breve{\mathcal{D}}_1(\varepsilon)$$
$$+\mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{D}}_s(\varepsilon)\,, \tag{48}$$

$$\frac{d}{d\varepsilon}\breve{\mathcal{D}}_i(\varepsilon) = \sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\breve{\mathcal{D}}_i(\varepsilon) \tag{49}$$
$$-A^T(\varepsilon)\breve{\mathcal{D}}_i(\varepsilon) + \mathcal{H}_i(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)\,,$$
$$\frac{d}{d\varepsilon}\mathcal{D}_1(\varepsilon) = -\mathrm{Tr}\left\{\mathcal{H}_1(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\} - z^T(\varepsilon)Q(\varepsilon)z(\varepsilon)$$
$$+2\breve{\mathcal{D}}_1^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)$$
$$-\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{s=1}^{k}\widehat{\mu}_s\breve{\mathcal{D}}_s(\varepsilon)\,, \tag{50}$$
$$\frac{d}{d\varepsilon}\mathcal{D}_i(\varepsilon) = -\mathrm{Tr}\left\{\mathcal{H}_i(\varepsilon)G(\varepsilon)WG^T(\varepsilon)\right\}$$
$$+2\breve{\mathcal{D}}_i^T(\varepsilon)B(\varepsilon)R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r(\varepsilon)\,. \tag{51}$$

where the terminal conditions $\mathcal{H}_1(t_f) = C^T(t_f)Q_fC(t_f)$, $\mathcal{H}_i(t_f) = 0$ for $2 \le i \le k$; $\breve{\mathcal{D}}_1(t_f) = -C^T(t_f)Q_fz(t_f)$, $\breve{\mathcal{D}}_i(t_f) = 0$ for $2 \le i \le k$ and $\mathcal{D}_1(t_f) = z^T(t_f)Q_fz(t_f)$, $\mathcal{D}_i(t_f) = 0$ for $2 \le i \le k$. The boundary condition of $\mathcal{W}(\varepsilon, \mathcal{Y}, \breve{\mathcal{Z}}, \mathcal{Z})$ implies that

$$x_0^T\sum_{i=1}^{k}\mu_i\left(\mathcal{H}_{i0} + \mathcal{E}_i(t_0)\right)x_0$$
$$+2x_0^T\sum_{i=1}^{k}\mu_i\left(\breve{\mathcal{D}}_{i0} + \breve{\mathcal{T}}_i(t_0)\right) + \sum_{i=1}^{k}\mu_i\left(\mathcal{D}_{i0} + \mathcal{T}_i(t_0)\right)$$
$$= x_0^T\sum_{i=1}^{k}\mu_i\mathcal{H}_{i0}x_0 + 2x_0^T\sum_{i=1}^{k}\mu_i\breve{\mathcal{D}}_{i0} + \sum_{i=1}^{k}\mu_i\mathcal{D}_{i0}\,.$$

The initial conditions for the equations (40)-(45) follow $\mathcal{E}_i(t_0) = 0$, $\breve{\mathcal{T}}_i(t_0) = 0$, and $\mathcal{T}_i(t_0) = 0$. Therefore, the optimal external signal (37) and state-feedback feedback gain (38) minimizing the performance index (26) become

$$u_{ext}^*(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}_r^*(\varepsilon)\,,$$
$$K^*(\varepsilon) = -R^{-1}(\varepsilon)B^T(\varepsilon)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}_r^*(\varepsilon)\,.$$

The theorem that follows contains a controller design algorithm which is able to track a prescribed function of time in the optimal kCC sense. The optimal kCC tracking controller requires a standard state-feedback kCC control design and an additional signal that results from the backward solutions of linear differential equations.

Theorem 5: (Finite-Horizon kCC Control Solution for Tracking Problems)
Let $A \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times n})$, $B \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times m})$, $C \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r\times n})$, and $G \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{n\times p})$. The tracking problem is then described by the equations (1)-(2) where the input noise $w(t) \in \mathbb{R}^p$ is the p-dimensional Wiener process starting from t_0, independent of the initial condition x_0, and defined on a complete probability space

$(\Omega, \mathcal{F}, \mathcal{P})$ over $[t_0, t_f]$ with the correlation of increments $E\left\{[w(\tau) - w(\xi)][w(\tau) - w(\xi)]^T\right\} = W|\tau - \xi|$ and $W > 0$. The control input $u \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^m))$ to the specified system is selected so that the resulting output $y \in L^2_{\mathcal{F}_t}(\Omega; \mathcal{C}([t_0, t_f]; \mathbb{R}^r))$ best approximates the known a priori trajectory $z \in L^2(\mathcal{C}([t_0, t_f]; \mathbb{R}^r))$ in the sense of (26) in which the terminal penalty error weighting $Q_f \in \mathbb{R}^{r \times r}$, the error weighting $Q \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{r \times r})$, and the control input weighting $R \in \mathcal{C}([t_0, t_f]; \mathbb{R}^{m \times m})$ are symmetric and positive semidefinite with $R(t)$ invertible.

Assume both $k \in \mathbb{Z}^+$ and the sequence $\mu = \{\mu_i \geq 0\}_{i=1}^k$ with $\mu_1 > 0$ are fixed. Then, the linear state-feedback kCC control solution for the finite-horizon tracking problem is implemented by

$$u^*(t) = K^*(t)x^*(t) + u^*_{ext}(t)\,, \tag{52}$$

$$K^*(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\mathcal{H}^*_r(\alpha)\,, \tag{53}$$

$$u^*_{ext}(\alpha) = -R^{-1}(\alpha)B^T(\alpha)\sum_{r=1}^{k}\widehat{\mu}_r\breve{\mathcal{D}}^*_r(\alpha)\,, \tag{54}$$

where $\widehat{\mu}_r = \mu_i/\mu_1$ and whenever $\{\mathcal{H}^*_r(\alpha)\}_{r=1}^k$, and $\left\{\breve{\mathcal{D}}^*_r(\alpha)\right\}_{r=1}^k$ are the solutions of the backward-in-time matrix differential equations

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}^*_1(\alpha) = &-\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]^T\mathcal{H}^*_1(\alpha)\\ &-\mathcal{H}^*_1(\alpha)\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]\\ &-C^T(\alpha)Q(\alpha)C(\alpha) - K^{*T}(\alpha)R(\alpha)K^*(\alpha)\,,\end{aligned} \tag{55}$$

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{H}^*_r(\alpha) = &-\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]^T\mathcal{H}^*_r(\alpha)\\ &-\mathcal{H}^*_r(\alpha)\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]\\ &-\sum_{s=1}^{r-1}\frac{2r!}{s!(r-s)!}\mathcal{H}^*_s(\alpha)G(\alpha)WG^T(\alpha)\mathcal{H}^*_{r-s}(\alpha),\end{aligned} \tag{56}$$

and the backward-in-time vector differential equations

$$\begin{aligned}\frac{d}{d\alpha}\breve{\mathcal{D}}^*_1(\alpha) = &-\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]^T\breve{\mathcal{D}}^*_1(\alpha)\\ &-\mathcal{H}_1(\alpha)B(\alpha)u^*_{ext}(\alpha)\\ &-K^{*T}(\alpha)R(\alpha)u^*_{ext}(\alpha) + C^T(\alpha)Q(\alpha)z(\alpha)\,,\end{aligned} \tag{57}$$

$$\begin{aligned}\frac{d}{d\alpha}\breve{\mathcal{D}}^*_r(\alpha) = &-\left[A(\alpha) + B(\alpha)K^*(\alpha)\right]^T\breve{\mathcal{D}}^*_r(\alpha)\\ &-\mathcal{H}_r(\alpha)B(\alpha)u^*_{ext}(\alpha)\end{aligned} \tag{58}$$

with the terminal boundary conditions $\mathcal{H}^*_1(t_f) = C^T(t_f)Q_fC(t_f)$, $\mathcal{H}^*_r(t_f) = 0$ for $2 \leq r \leq k$ and $\breve{\mathcal{D}}^*_1(t_f) = -C^T(t_f)Q_fz(t_f)$, $\breve{\mathcal{D}}^*_r(t_f) = 0$ for $2 \leq r \leq k$.

IV. Conclusions

In this paper, an optimal control problem for a wide class of tracking systems is formulated in which the objective is minimization of a finite, linear combination of cumulants of integral quadratic cost over linear, memoryless, full-state-feedback control laws. The standard linear tracking system constraint on a finite time interval with additive Wiener noise and a non-random initial state underlies the problem formulation. Because of the linearity assumptions in the problem statement, it can be formulated as a non-stochastic optimization problem utilizing equations for cost cumulants developed in this exposition. Furthermore, this problem formulation is parameterized both by the number of cumulants and by the scalar coefficients in the linear combination, it defines a very general Linear-Quadratic-Gaussian (LQG) and Risk Sensitive problem classes. The special cases where only the first cost cumulant is minimized is, of course, the well known minimum mean LQG problem and whereas a denumerable linear combination of cost cumulants is minimized is the continued Risk Sensitive control objective. It should also be noted that although the optimization criterion of the cost-cumulant control problem represents a competition among cumulant values, the ultimate objective herein is to introduce parametric freedom in the class of feedback control laws which will result from the problem solution. This parametric freedom has been exploited to achieve desirable closed-loop system properties as illustrated in [1]-[5]. Finally, the general solution of the cost-cumulant control problem for the class of linear-quadratic tracking systems is presented and is determined by a feedback cost-cumulant control obtained by a set of coupled Riccati-type differential equations and time-dependent tracking variables found by solving an auxiliary set of coupled differential equations (incorporating the desired trajectory) backward from a stable final time. The issue of existence of solution to the optimization problem becomes that of existence of solutions to the Riccati-type equations. Conditions ensuring existence of solutions to these equations are being worked out. In fact, for values of linear combination coefficients outside certain finite ranges, the equations exhibit finite escape time behavior. On the other hand, for limited ranges of the combination coefficient values, the equations are well behaved and yield steady-state solutions as shown in various controller designs [1]-[5].

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